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On a quaternionic Maxwell equation for the time-dependent electromagnetic field in a chiral medium

Sergei M Grudsky[1], **Kira V Khmelnytskaya**[2] **and Vladislav V Kravchenko**[2]

[1] Department of Mathematics, CINVESTAV, National Polytechnic Institute, Mexico City, Mexico
[2] Department of Telecommunications, SEPI ESIME Zacatenco, National Polytechnic Institute, Av. IPN S/N, C.P.07738 D.F., Mexico

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Abstract
Maxwell's equations for the time-dependent electromagnetic field in a homogeneous chiral medium are reduced to a single quaternionic equation. Its fundamental solution satisfying the causality principle is obtained which allows us to solve the time-dependent chiral Maxwell system with sources.

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1. Introduction

We consider Maxwell's equations for the time-dependent electromagnetic field in a homogeneous chiral medium and show their equivalence to a single quaternionic equation. This result generalizes the well-known (see [9, 16, 23]) quaternionic reformulation of the Maxwell equations for non-chiral media. Nevertheless, the new quaternionic differential operator is essentially different from the quaternionic operator corresponding to the non-chiral case. We obtain a fundamental solution of the new operator in explicit form satisfying the causality principle. Its convolution with a quaternionic function representing sources of the electromagnetic field gives us a solution of the inhomogeneous Maxwell system in a whole space.

2. Maxwell's equations for chiral media

Consider time-dependent Maxwell's equations

$$\operatorname{rot} \overrightarrow{E}(t,x) = -\partial_t \overrightarrow{B}(t,x) \tag{1}$$

$$\operatorname{rot} \overrightarrow{H}(t,x) = \partial_t \overrightarrow{D}(t,x) + \overrightarrow{j}(t,x) \tag{2}$$

$$\operatorname{div}\overrightarrow{E}(t,x)=\frac{\rho(t,x)}{\varepsilon},\qquad \operatorname{div}\overrightarrow{H}(t,x)=0 \tag{3}$$

with the Drude–Born–Fedorov constitutive relations corresponding to the chiral media [3, 20, 21]

$$\overrightarrow{B}(t,x)=\mu(\overrightarrow{H}(t,x)+\beta\operatorname{rot}\overrightarrow{H}(t,x)) \tag{4}$$

$$\overrightarrow{D}(t,x)=\varepsilon(\overrightarrow{E}(t,x)+\beta\operatorname{rot}\overrightarrow{E}(t,x)) \tag{5}$$

where β is the chirality measure of the medium. β, ε, μ are real scalars assumed to be constants. Note that the charge density ρ and the current density $\overrightarrow{j}$ are related by the continuity equation $\partial_t\rho+\operatorname{div}\overrightarrow{j}=0$.

Incorporating the constitutive relations (4) and (5) into the system (1)–(3) we arrive at the main object of our study, the time-dependent Maxwell system for a homogeneous chiral medium

$$\operatorname{rot}\overrightarrow{H}(t,x)=\varepsilon(\partial_t\overrightarrow{E}(t,x)+\beta\partial_t\operatorname{rot}\overrightarrow{E}(t,x))+\overrightarrow{j}(t,x) \tag{6}$$

$$\operatorname{rot}\overrightarrow{E}(t,x)=-\mu(\partial_t\overrightarrow{H}(t,x)+\beta\partial_t\operatorname{rot}\overrightarrow{H}(t,x)) \tag{7}$$

$$\operatorname{div}\overrightarrow{E}(t,x)=\frac{\rho(t,x)}{\varepsilon}\qquad \operatorname{div}\overrightarrow{H}(t,x)=0. \tag{8}$$

Application of rot to (6) and (7) allows us to separate the equations for $\overrightarrow{E}$ and $\overrightarrow{H}$ and obtain in this way the wave equations for a chiral medium

$$\operatorname{rot}\operatorname{rot}\overrightarrow{E}+\varepsilon\mu\partial_t^2\overrightarrow{E}+2\beta\varepsilon\mu\partial_t^2\operatorname{rot}\overrightarrow{E}+\beta^2\varepsilon\mu\partial_t^2\operatorname{rot}\operatorname{rot}\overrightarrow{E}=-\mu\partial_t\overrightarrow{j}-\beta\mu\partial_t\operatorname{rot}\overrightarrow{j} \tag{9}$$

$$\operatorname{rot}\operatorname{rot}\overrightarrow{H}+\varepsilon\mu\partial_t^2\overrightarrow{H}+2\beta\varepsilon\mu\partial_t^2\operatorname{rot}\overrightarrow{H}+\beta^2\varepsilon\mu\partial_t^2\operatorname{rot}\operatorname{rot}\overrightarrow{H}=\operatorname{rot}\overrightarrow{j}. \tag{10}$$

It should be noted that when $\beta=0$, (9) and (10) reduce to the usual second order wave equations for non-chiral media. When $\beta\neq 0$ the corresponding wave equations represent partial differential equations of fourth order.

3. Some notation from quaternionic analysis

We will consider biquaternion-valued functions defined in some domain $\Omega\subset\mathbb{R}^3$. On the set of such continuously differentiable functions the well-known Moisil–Teodoresco operator is defined by the expression $D=i_1\frac{\partial}{\partial x_1}+i_2\frac{\partial}{\partial x_2}+i_3\frac{\partial}{\partial x_3}$ (see, e.g., [8]), where i_k, $k=1,2,3$, are basic quaternionic imaginary units. Denote $D_\alpha=D+\alpha$, where $\alpha\in\mathbb{C}$ and $\operatorname{Im}\alpha\geqslant 0$. The fundamental solution for this operator is known [14] (see also [16]):

$$\mathcal{K}_\alpha(x)=-\operatorname{grad}\Theta_\alpha(x)+\alpha\Theta_\alpha(x)=\left(\alpha+\frac{x}{|x|^2}-\mathrm{i}\alpha\frac{x}{|x|}\right)\Theta_\alpha(x) \tag{11}$$

where i is the usual complex imaginary unit commuting with i_k, $x=\sum_{k=1}^{3}x_ki_k$ and $\Theta_\alpha(x)=-\frac{\mathrm{e}^{\mathrm{i}\alpha|x|}}{4\pi|x|}$. Note that besides the equation $D_\alpha\mathcal{K}_\alpha=\delta$ where δ is the Dirac delta function, $\mathcal{K}_\alpha$ fulfils the following radiation condition at infinity uniformly in all directions:

$$\left(1+\frac{\mathrm{i}x}{|x|}\right)\cdot\mathcal{K}_\alpha(x)=o\left(\frac{1}{|x|}\right)\qquad\text{when } |x|\to\infty \tag{12}$$

which is in agreement with the Silver–Müller radiation conditions [12].

4. Field equations in quaternionic form

In this section we rewrite the field equations from section 2 in quaternionic form.

Let us introduce the following quaternionic operator:

$$M = \beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t - \mathrm{i}D \tag{13}$$

and consider the purely vectorial biquaternionic function

$$\overrightarrow{V}(t,x) = \overrightarrow{E}(t,x) - \mathrm{i}\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{H}(t,x). \tag{14}$$

Proposition 1. *The quaternionic equation*

$$M\overrightarrow{V}(t,x) = -\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{j}(t,x) - \beta\sqrt{\frac{\mu}{\varepsilon}}\partial_t\rho(t,x) + \frac{\mathrm{i}\rho(t,x)}{\varepsilon} \tag{15}$$

is equivalent to the Maxwell system (6)–(8), the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ are solutions of (6)–(8) if and only if the purely vectorial biquaternionic function $\overrightarrow{V}$ defined by (14) is a solution of (15).

Proof. The scalar and vector parts of (15) have the form

$$-\beta\sqrt{\varepsilon\mu}\partial_t \operatorname{div}\overrightarrow{E} + \sqrt{\frac{\mu}{\varepsilon}}\operatorname{div}\overrightarrow{H} + \mathrm{i}(\operatorname{div}\overrightarrow{E} + \beta\mu\partial_t\operatorname{div}\overrightarrow{H}) = -\beta\sqrt{\frac{\mu}{\varepsilon}}\partial_t\rho + \frac{\mathrm{i}\rho}{\varepsilon} \tag{16}$$

$$\beta\sqrt{\varepsilon\mu}\partial_t \operatorname{rot}\overrightarrow{E} + \sqrt{\varepsilon\mu}\partial_t\overrightarrow{E} - \sqrt{\frac{\mu}{\varepsilon}}\operatorname{rot}\overrightarrow{H} - \mathrm{i}(\operatorname{rot}\overrightarrow{E} + \beta\mu\partial_t\operatorname{rot}\overrightarrow{H} + \mu\partial_t\overrightarrow{H}) = -\sqrt{\frac{\mu}{\varepsilon}}\overrightarrow{j}. \tag{17}$$

The real part of (17) coincides with (6) and the imaginary part coincides with (7). Applying divergence to equation (17) and using the continuity equation gives us

$$\partial_t \operatorname{div}\overrightarrow{H} = 0 \qquad \text{and} \qquad \partial_t \operatorname{div}\overrightarrow{E} = \frac{1}{\varepsilon}\partial_t\rho.$$

Taking into account these two equalities, we obtain from (16) that the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ satisfy equations (8). □

It should be noted that for $\beta = 0$ from (13) we obtain the operator which was studied in [11] with the aid of the factorization of the wave operator for non-chiral media

$$\varepsilon\mu\partial_t^2 - \Delta_x = (\sqrt{\varepsilon\mu}\partial_t + \mathrm{i}D)(\sqrt{\varepsilon\mu}\partial_t - \mathrm{i}D).$$

In the case under consideration, we obtain a similar result. Let us denote by M^* the complex conjugate operator of M:

$$M^* = \beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t + \mathrm{i}D.$$

For simplicity we now consider a sourceless situation. In this case equations (9) and (10) are homogeneous and can be represented as follows:

$$MM^*\overrightarrow{U}(t,x) = 0$$

where $\overrightarrow{U}$ stands for $\overrightarrow{E}$ or for $\overrightarrow{H}$.

5. Fundamental solution of the operator *M*

We will construct a fundamental solution of the operator M using the results of the previous section and well-known facts from quaternionic analysis. Consider the equation

$$(\beta\sqrt{\varepsilon\mu}\partial_t D + \sqrt{\varepsilon\mu}\partial_t - \mathrm{i}D)f(t,x) = \delta(t,x).$$

Applying the Fourier transform $\mathcal{F}$ with respect to the time variable t, we obtain

$$(\beta\sqrt{\varepsilon\mu}\mathrm{i}\omega D + \sqrt{\varepsilon\mu}\mathrm{i}\omega - \mathrm{i}D)F(\omega,x) = \delta(x)$$

where $F(\omega,x) = \mathcal{F}\{f(t,x)\} = \int_{-\infty}^{\infty} f(t,x)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$. The last equation can be rewritten as follows:

$$(D+\alpha)(\beta\sqrt{\varepsilon\mu}\omega - 1)\,\mathrm{i}F(\omega,x) = \delta(x)$$

where $\alpha = \frac{\sqrt{\varepsilon\mu}\omega}{\beta\sqrt{\varepsilon\mu}\omega-1}$. The fundamental solution of D_α is given by (11), so we have

$$(\beta\sqrt{\varepsilon\mu}\omega - 1)\,\mathrm{i}F(\omega,x) = \mathcal{K}_\alpha(x) = \left(\alpha + \frac{x}{|x|^2} - \mathrm{i}\alpha\frac{x}{|x|}\right)\Theta_\alpha(x)$$

from where

$$F(\omega,x) = \left[\frac{\mathrm{i}\sqrt{\varepsilon\mu}\omega}{(\beta\sqrt{\varepsilon\mu}\omega-1)^2}\left(1 - \frac{ix}{|x|}\right) + \frac{\mathrm{i}x}{|x|^2}\frac{1}{\beta\sqrt{\varepsilon\mu}\omega - 1}\right]\frac{\mathrm{e}^{\mathrm{i}|x|\frac{\sqrt{\varepsilon\mu}\omega}{\beta\sqrt{\varepsilon\mu}\omega-1}}}{4\pi|x|}.$$

We write it in a more convenient form

$$F(\omega,x) = \left(\frac{1}{(\omega-a)^2}A(x) + \frac{1}{\omega-a}B(x)\right)E(x)\,\mathrm{e}^{\frac{\mathrm{i}c(x)}{\omega-a}}$$

where

$$a = \frac{1}{\beta\sqrt{\varepsilon\mu}} \qquad c(x) = \frac{|x|}{\beta^2\sqrt{\varepsilon\mu}} \qquad E(x) = \frac{\mathrm{e}^{\frac{\mathrm{i}|x|}{\beta}}}{4\pi|x|}$$

$$A(x) = \frac{\mathrm{i}}{\beta^3\varepsilon\mu}\left(1 - \frac{\mathrm{i}x}{|x|}\right) \qquad B(x) = \frac{\mathrm{i}}{\beta\sqrt{\varepsilon\mu}}\left(\frac{1}{\beta}\left(1 - \frac{\mathrm{i}x}{|x|}\right) + \frac{x}{|x|^2}\right).$$

In order to obtain the fundamental solution $f(t,x)$ we should apply the inverse Fourier transform to $F(\omega,x)$. Among different regularizations of the resulting integral, we should choose the one leading to a fundamental solution satisfying the causality principle, that is vanishing for $t < 0$. Such election is done by introducing a small parameter $y > 0$ in the following way:

$$f(t,x) = \lim_{y\to 0}\mathcal{F}^{-1}\{F(z,x)\} \tag{18}$$

where $z = \omega - \mathrm{i}y$. This regularization is in agreement with the condition $\operatorname{Im}\alpha \geqslant 0$. We have

$$\mathcal{F}^{-1}\{F(z,x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(\omega-a_y)^2}A(x) + \frac{1}{\omega-a_y}B(x)\right)E(x)\,\mathrm{e}^{\frac{\mathrm{i}c(x)}{\omega-a_y}}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{19}$$

where $a_y = a + \mathrm{i}y$. Expression (19) includes two integrals of the form

$$I_k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\frac{\mathrm{i}c}{\omega-a_y}}\,\mathrm{e}^{\mathrm{i}\omega t}}{(\omega-a_y)^k}\,\mathrm{d}\omega, \qquad k = 1, 2$$

where $c = c(x)$. We have

$$I_k = \frac{1}{2\pi}\sum_{j=0}^{\infty}\left(\frac{(\mathrm{i}c)^j}{j!}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega}{(\omega-a_y)^{j+k}}\right) \tag{20}$$

where the change of order of integration and summation is possible because the two necessary conditions are fulfilled: the series is uniformly convergent on each segment and the integrals of partial sums converge uniformly with respect to j. Denote

$$I_{k,j}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega}{(\omega - a_y)^{j+k}}.$$

For $k = 1$ and $j = 0$ we obtain (see, e.g., [5, section 8.7])

$$I_{1,0}(t) = 2\pi\,\mathrm{i}H(t)\,\mathrm{e}^{\mathrm{i}ta_y}$$

where H is the Heaviside function. For all other cases, that is for $k = 1$ and $j = \overline{1,\infty}$ and for $k = 2$ and $j = \overline{0,\infty}$, we have that $j + k \geqslant 2$ and the integrand in (20) has a pole at the point a_y of order $j + k$. Using a result from the residue theory [6, section 4.3], we obtain

$$I_{k,j}(t) = 2\pi\mathrm{i}\operatorname*{Res}_{a_y} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{(\omega - a_y)^{j+k}} \qquad \text{for } t \geqslant 0 \quad \text{and} \quad j + k \geqslant 2.$$

Consider

$$\operatorname*{Res}_{a_y} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{(\omega - a_y)^{j+k}} = \frac{1}{(j+k-1)!}\lim_{\omega\to a_y} \frac{\partial^{j+k-1}}{\partial\omega^{j+k-1}}\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{(\mathrm{i}t)^{j+k-1}\,\mathrm{e}^{\mathrm{i}a_y t}}{(j+k-1)!}$$
$$\text{for} \quad t \geqslant 0 \quad \text{and } j + k \geqslant 2.$$

For $t < 0$ we have that $I_{k,j}(t)$ is equal to the sum of residues with respect to singularities in the lower half-plane $y < 0$ which is zero because the integrand is analytic there. Thus we obtain

$$I_{k,j}(t) = 2\pi\,\mathrm{i}H(t)\frac{(\mathrm{i}t)^{j+k-1}}{(j+k-1)!}\,\mathrm{e}^{\mathrm{i}a_y t}.$$

Substitution of this result into (20) gives us

$$I_1 = \mathrm{i}H(t)\,\mathrm{e}^{\mathrm{i}a_y t}\sum_{j=0}^{\infty}\frac{(-ct)^j}{j!j!} \qquad \text{and} \qquad I_2 = -H(t)\,\mathrm{e}^{\mathrm{i}a_y t}\,t\sum_{j=0}^{\infty}\frac{(-ct)^j}{j!(j+1)!}.$$

Now using the series representations of the Bessel functions J_0 and J_1 (see e.g. [24, chapter 5]), we obtain

$$I_1 = \mathrm{i}H(t)\,\mathrm{e}^{\mathrm{i}a_y t}J_0(2\sqrt{ct}) \qquad \text{and} \qquad I_2 = -H(t)\sqrt{\frac{t}{c}}\,\mathrm{e}^{\mathrm{i}a_y t}J_1(2\sqrt{ct}).$$

Substituting these expressions in (19) and then in (18), we arrive at the following expression for f:

$$f(t,x) = H(t)\,\mathrm{e}^{\mathrm{i}at}E(x)\left(-A(x)\sqrt{\frac{t}{c}}J_1(2\sqrt{ct}) + \mathrm{i}B(x)J_0(2\sqrt{ct})\right).$$

Finally we rewrite the obtained fundamental solution of the operator M in explicit form:

$$f(t,x) = H(t)\frac{\mathrm{e}^{\frac{\mathrm{i}t}{\beta\sqrt{\varepsilon\mu}}}}{\beta\sqrt{\varepsilon\mu}}\left(\mathcal{K}_{\frac{1}{\beta}}(x)J_0\left(\frac{2\sqrt{t|x|}}{\beta(\varepsilon\mu)^{\frac{1}{4}}}\right) + \frac{\mathrm{i}\Theta_{\frac{1}{\beta}}(x)}{\beta(\varepsilon\mu)^{\frac{1}{4}}}\left(1 - \frac{ix}{|x|}\right)\sqrt{\frac{t}{|x|}}J_1\left(\frac{2\sqrt{t|x|}}{\beta(\varepsilon\mu)^{\frac{1}{4}}}\right)\right).$$

Let us note that f fulfils the causality principle requirement which guarantees that its convolution with the function on the right-hand side of (15) gives us the unique physically meaningful solution of the inhomogeneous Maxwell system (6)–(8) in whole space.

6. Concluding remarks

We have shown that the Maxwell system for the time-dependent electromagnetic field in a homogeneous chiral medium is equivalent to a single quaternionic equation. As a first natural step in studying the new partial differential operator, we obtained the fundamental solution of the quaternionic Maxwell operator for chiral media satisfying the causality principle. This allows us to solve the inhomogeneous Maxwell system in a whole space.

Quaternionic analysis methods proved to be powerful and necessary for solving a wide spectrum of problems for the electromagnetic field such as boundary value problems (see, e.g., [4, 8, 10, 12, 16–18, 22]), construction of exact and asymptotic solutions (see, e.g., [1, 13, 16]), analysis of interesting physical implications of quaternionic models (see, e.g., [7, 15, 19]) and many others. The quaternionic reformulation of Maxwell's equations for the time-dependent electromagnetic field in a chiral medium obtained in this work opens the way for applications of quaternionic analysis technique to this important physical model.

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